

Topics for Phd program: Algebra

Calculus

1. **Limits:** The concept of a limit helps us understand the behavior of functions as they approach certain points, including infinity. It's foundational to calculus.
2. **Derivatives:** This topic involves studying the rate of change of a function, or how a function changes as its input changes. It's central to concepts like motion and optimization.
3. **Applications of Derivatives:** Includes understanding concepts like tangent lines, optimization problems, and related rates, where derivatives are applied to real-world situations.
4. **Integrals:** The integral is the reverse process of differentiation and is used to calculate areas under curves, volumes, and accumulated quantities.
5. **Taylor's formula:** Taylor's polynomials. The remainder in various forms: integral, Peano etc.
6. **Fundamental Theorem of Calculus:** This theorem links the concept of the derivative with the integral, stating that differentiation and integration are inverse operations.
7. **Techniques of Integration:** Various methods to compute integrals, such as substitution, integration by parts, partial fractions, and trigonometric identities.
8. **Infinite Sequences and Series:** Understanding the behavior of sequences and series, including concepts like convergence, divergence, and power series.
9. **Multivariable Calculus:** This extends calculus to functions of more than one variable, covering topics like partial derivatives, multiple integrals, and gradient fields.
10. **Vector Calculus:** Deals with vector fields, line integrals, surface integrals, and theorems like Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

Reading materials:

- Thomas' Calculus Early Transcendentals, Twelfth Edition. We'll cover most of chapters 1 – 8.
- James Stewart's Calculus, 7th edition
- Frank Ayres, Elliot Mendelson, Calculus, 4th edition, Schaum's Outlines.
- Howard Anton, Calculus, John Wiley and Sons / New York – 1999.
- C. Henry Edwards, David E. Penney, Calculus (6th Edition)
- Robert Adams, Calculus A Complete Course, Pearson, Toronto, 2006
- Jon Rogawski, Calculus

10 main topics in Real Analysis:

1. **Sequences and Limits:** The supremum and infimum of sequences of numbers and sets, convergence of Cauchy sequences, and the properties of limit points.
2. **Uniform Continuity:** A function is uniformly continuous if small changes in the input lead to small changes in the output, uniformly across the entire domain. A detailed exploration of uniformly continuous functions, including the Lipschitz continuity.
3. **Topology of the Real Line:** Understanding open, closed, and compact sets, connectedness, and the structure of the real number line.
4. **Metric Spaces:** Generalizing distance, studying spaces equipped with a metric, and exploring concepts such as completeness and compactness in these spaces.
5. **Compactness:** Understanding the important topological concept of compact sets, which are closed and bounded in Euclidean spaces and have special properties like every sequence having a convergent subsequence.
6. **Sequences and Series of Functions:** Investigating pointwise and uniform convergence, and the interaction between limits and operations like integration and differentiation.
7. **Convergence Theorems:** The Dominated Convergence Theorem, Fatou's Lemma, and other important theorems for the interchange of limits and integrals.
8. **The Baire Category Theorem:** A foundational result in topology and analysis that describes the structure of complete metric spaces and their applications in real analysis.
9. **Lebesgue Measure and Integration (Introduction):** The basic ideas behind Lebesgue integration, an alternative to Riemann integration that allows for a broader class of integrable functions.
10. **Implicit and Inverse Function Theorems:** The conditions under which a system of equations defines a function implicitly and the properties of such functions. Describes conditions under which a differentiable function has a differentiable inverse, and the relationship between their derivatives.

Reading materials:

- Robert G. Bartle, Introduction to Real Analysis
- Stephen Abbott, Understanding Analysis

Algebra

1. **Binary Operations:** Definition of a binary operation on a set. Properties: closure, associativity, commutativity, identity elements. Examples from arithmetic operations and custom-defined operations.

- 2. Groups:** Formal definition using axioms. Abelian vs non-abelian groups. Basic notation, order of elements, and introduction to structure.
- 3. Examples of Groups:** Integers under addition, nonzero reals under multiplication, modular groups \mathbb{Z}_n . Introduction to matrix groups and permutation groups.
- 4. Elementary Properties of Groups:** Cancellation laws, solving equations in groups. Properties involving inverses and identity. Basic manipulation techniques.
- 5. Uniqueness of Identity and Inverses:** Proof of uniqueness of identity element. Proof that each element has a unique inverse. Consequences for group structure.
- 6. Finite Groups and Group Tables:** Definition of order of a group and elements. Construction of Cayley tables. Identifying identity, inverses, and structure from tables.
- 7. Isomorphism of Groups:** Definition of isomorphism and automorphism. Structural equivalence of groups. Invariants such as order and cyclicity.
- 8. Subgroups:** Definition and subgroup tests (closure test, subgroup criterion). Trivial and improper subgroups. Intersection of subgroups.
- 9. Examples of Subgroups:** Subgroups of \mathbb{Z} , cyclic groups, and symmetric groups. Generated subgroups and examples in practice.
- 10. Subgroup Diagram:** Lattice of subgroups. Inclusion relations. Examples for small finite groups like \mathbb{Z}_n and S_3 .
- 11. Cyclic Subgroups:** Generation by a single element. Powers of elements. Order of an element and relation to subgroup size.
- 12. Cyclic Groups:** Definition and generators. Infinite cyclic group vs finite cyclic groups. Structure and classification.
- 13. Examples and Properties of Cyclic Groups:** Examples like \mathbb{Z} , \mathbb{Z}_n . Every subgroup is cyclic. Relation between generators and order.
- 14. Permutation Groups:** Definition of permutations and symmetric group S_n . Composition and inverses. Applications in symmetry.
- 15. Orbits:** Definition via group actions. Orbit of an element and orbit decomposition. Basic orbit-stabilizer ideas.
- 16. Cycles:** Cycle notation and disjoint cycle decomposition. Length of cycles. Simplification of permutation computations.

- 17. Even and Odd Permutations:** Definition via number of transpositions. Sign of a permutation. Properties of parity.
- 18. Alternating Group:** Definition of A_n . Structure and order. Relation to symmetric group and normality.
- 19. Cosets and Lagrange's Theorem:** Definition of left and right cosets. Index of a subgroup. Statement and proof of Lagrange's theorem and applications.
- 20. Homomorphisms of Groups:** Definition and examples. Preservation of operation. Types: injective, surjective, bijective homomorphisms.
- 21. Kernel and Image:** Definition and properties. Kernel as a normal subgroup. Image as a subgroup of the codomain.
- 22. Direct Product of Groups:** Construction of product groups. Componentwise operations. Examples and properties (abelian case, orders).
- 23. Finitely Generated Abelian Groups:** Definition of generators. Structure theorem statement. Decomposition into cyclic groups and invariants.
- 24. Factor Groups:** Definition using normal subgroups. Construction of quotient groups. Examples and computations.
- 25. Fundamental Homomorphism Theorem:** Statement and proof idea. Relationship between kernel and quotient. Applications in simplifying groups.
- 26. Isomorphism Theorems:** First, second, and third isomorphism theorems. Relationships between subgroups and quotients. Applications in structure analysis.
- 27. Rings and Fields:** Definition of rings (with unity, commutative/non-commutative). Definition of fields. Examples and comparison.
- 28. Integral Domains:** Definition as commutative rings without zero divisors. Cancellation property. Examples and non-examples.
- 29. Fermat's and Euler's Theorems:** Statements in modular arithmetic. Conditions (prime modulus, coprime integers). Applications in computations and cryptography.
- 30. Field of Quotients:** Construction of fractions from an integral domain. Embedding into a field. Example: integers \rightarrow rationals.
- 31. Rings of Polynomials:** Definition over a ring/field. Degree, leading coefficient. Addition, multiplication, and division (over fields).

32. Factorization of Polynomials: Irreducibility criteria. Factorization over different fields. Examples and basic techniques.

33. Ring Homomorphisms and Factor Rings: Definition of ring homomorphisms. Ideals and quotient rings. Correspondence between kernels and ideals.

34. Prime and Maximal Ideals: Definitions and properties. Characterization via quotient rings. Examples in polynomial rings and integers.

35. Unique Factorization Domains (UFD): Definition and uniqueness of factorization. Examples like \mathbb{Z} and polynomial rings. Relation to irreducibles and primes.

36. Euclidean Domains: Definition via Euclidean function. Division algorithm. Examples and implication: Euclidean \Rightarrow PID \Rightarrow UFD.

37. Gaussian Integers: Definition $a + bi$. Arithmetic operations and norm. Divisibility, units, and factorization properties.

Reading materials:

- Fraleigh John B. A First Course in Abstract Algebra Seventh Edition, 2014, Pearson New International Edition.
- Linda L., Linda J. Elements of Modern Algebra, Brooks/Cole, Seventh Edition, 2009

Differential Geometry and topology:

1. **Curves and Surfaces:** The study of the geometry of curves (1-dimensional objects) and surfaces (2-dimensional objects) in higher-dimensional spaces. Key concepts include arc length, curvature, and parametrization.
2. **Differential Manifolds:** A generalization of curves and surfaces to higher dimensions. These are spaces that locally resemble Euclidean space and are equipped with a smooth structure, allowing for calculus to be done on them.
3. **Tensors and Tensor Fields:** The study of tensors, which generalize vectors and matrices, and tensor fields, which describe geometric quantities like curvature, stress, and deformation on manifolds.
4. **Connections and Covariant Derivatives:** Connections provide a way of comparing vectors in different tangent spaces on a manifold. The covariant derivative measures how a vector field changes along the manifold.
5. **Riemannian Geometry:** Focuses on the study of smooth manifolds equipped with a Riemannian metric, which allows for measuring distances and angles. Key topics include geodesics (shortest paths), curvature, and the curvature tensor.
6. **Curvature:** A central concept in differential geometry, curvature describes how a curve or surface deviates from being flat. This includes Gaussian curvature (for surfaces) and sectional curvature (for higher-dimensional manifolds).

7. **Geodesics:** The "straight lines" on a curved surface or manifold, which are the shortest paths between points. Geodesics play a key role in understanding the intrinsic geometry of a manifold.
8. **The Gauss-Bonnet Theorem:** A fundamental result in differential geometry that connects the topology (the shape) of a surface to its geometry (the curvature). It relates the total curvature of a surface to its Euler characteristic.
9. **Lie Groups and Lie Algebras:** The study of smooth manifolds that also have a group structure, which is important in understanding symmetries and transformations. Lie groups are used in physics, particularly in the study of symmetry in space-time.
10. **Minimal Surfaces and Variational Problems:** The study of surfaces that locally minimize area, such as soap films. This topic connects differential geometry with variational calculus, where critical points of an energy functional represent minimal surfaces.

Reading materials:

- Andrew Pressley-Elementary Diff. Geom-2nd Ed-Springer, 2010
- Mishchenko, A. Fomenko, A Course Of Differential Geometry And Topology, 1988